THE GAUSS MAP FOR LAGRANGEAN AND ISOCLINIC SURFACES

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ABSTRACT. Simple properties of the Gauss map characterise important classes of surfaces in $\mathbf{R}^4$: R-surfaces, the real version of plane complex curves; Lagrangean surfaces; isoclinic surfaces.

1. INTRODUCTION

The origin of the study of surfaces in $\mathbf{R}^4$ was the interpretation of a complex plane curve as a real surface. In a different context, an invariant torus for a two degrees of freedom integrable Hamiltonian system is also a surface in $\mathbf{R}^4$.

These surfaces have very special properties, and our aim is to consider a bigger class of surfaces, containing all those, but retaining some of their properties, at least in a partial form.

If a surface is a real version of a complex plane curve, which we will call a R-surface [7], there are three properties upon which we will focus:

- All planes are isoclinic to each other
- The Gauss map has a constant component, in sphere coordinates.
- The surface is a Lagrangean submanifold of $\mathbf{R}^4$.

When considering two planes P and P' in $\mathbf{R}^4$, passing though the origin, we can take the angle that a unit vector u in P makes with its orthogonal projection u' in P'; when u describes circumference of radius 1 centred at the origin, that angle varies between two extreme values, in general different. The planes are said to be isoclinic to each other if in fact the angle remains constant.

This can be seen in different way: the unit circumference in P projects as an ellipse in P', and the axes correspond to the extreme

values of the angle above; thus the planes are isoclinic to each other when the ellipse is a circumference.

The Gauss map associates to a point in the surface its tangent plane seen as a vector subspace of $\mathbf{R}^4$; this is a point in the Grassmannian $G_{2,4}$. It is well known that $G_{2,4} \cong \mathbf{S}^2 \times \mathbf{S}^2$, thus the Gauss map is a pair of maps into spheres, and for a R-surface one of them is constant.

To define a Lagrangean submanifold we need to specify a symplectic structure, and in fact there are two natural choices for a R-surface, as will be seen in subsection 5.1.

We will consider a L-surface as a surface that is congruent to a Lagrangean surface with the standard symplectic form, and an I-surface, or isoclinic surface, a surface for which the Gauss curvature and the normal curvature coincide, up to sign. The I-surfaces can also be characterized for having an isoclinic tangent direction at every point [12].

Under very general hypotheses, given an initial curve, the Gauss map Γ allows the reconstruction of the surface which contains the curve and whose Gauss map is Γ [11]. This involves solving a partial differential equation.

Our aim here is just the characterization of the surface by simple properties of the Gauss map, and the main result is:

Theorem. *A surface $S \subset \mathbf{R}^4$ with Gauss map $\Gamma : S \longrightarrow G_{2,4} \cong \mathbf{S}^2 \times \mathbf{S}^2$ is a L-surface if, and only if, the image of Γ_1, or Γ_2, be contained in a great circle.*

2. BASIC DEFINITIONS AND RESULTS

We consider a surface $S \subset \mathbf{R}^4$ locally given by a parametrization:

$$\Xi : U \subset \mathbf{R}^2 \longrightarrow \mathbf{R}^4$$

and a set $\{e_1, e_2, e_3, e_4\}$ of orthonormal vectors, depending on $(x, y) \in U$, satisfying:

- $e_1(x, y)$ and $e_2(x, y)$ span the tangent space $T_{\Xi(x,y)}S$ of S at $\Xi(x, y)$.
- $e_3(x, y)$ and $e_4(x, y)$ span the normal space $N_{\Xi(x,y)}S$ of S at $\Xi(x, y)$.

Then $\Xi, \{e_1, e_2, e_3, e_4\}$ is an *adapted moving frame* for S.

While the image of $D\Xi$ is the tangent space of S, the image of the second derivative $D^2\Xi$ has both tangent and normal components; the vector valued quadratic form associated to the normal component:

$$(1) \qquad (D^2\Xi \cdot e_3)e_3 + (D^2\Xi \cdot e_4)e_4$$

is the *second fundamental form* **II** of S. It can be written [8] as $\mathbf{II}_1 e_3 + \mathbf{II}_2 e_4$, with:

$$(2a) \qquad \mathbf{II}_1 = au_1^2 + 2bu_1u_2 + cu_2^2$$

$$(2b) \qquad \mathbf{II}_2 = eu_1^2 + 2fu_1u_2 + gu_2^2$$

where $u = u_1 e_1 + u_2 e_2$ is a tangent vector.

We can express the Gaussian, normal and mean curvature in terms of the coefficients of the second fundamental form [8]:

$$(3) \qquad \begin{aligned} K &= (ac - b^2) + (eg - f^2) \\ \kappa &= (a - c)f - (e - g)b \\ \mathcal{H} &= \frac{1}{2}(a + c)e_3 + \frac{1}{2}(e + g)e_4 \end{aligned}$$

Assume the surface S locally given by a parametrization:

$$\Xi : (x, y) \mapsto (x, y, \varphi(x, y), \psi(x, y))$$

where $\Phi = (\varphi, \psi)$ has vanishing first jet at the origin, $j^1\Phi(0) = 0$.

The vectors T_1 and T_2 span the tangent space of S:

$$T_1 = \Xi_x = (1, 0, \varphi_x, \psi_x), \quad T_2 = \Xi_y = (0, 1, \varphi_y, \psi_y)$$

the index z standing for derivative with respect to z.

The induced metric in S is given by the first fundamental form:

$$\boldsymbol{I} = E\mathrm{d}x^2 + 2F\mathrm{d}x\mathrm{d}y + G\mathrm{d}y^2$$

where:

$$E = T_1 \cdot T_1, \quad F = T_1 \cdot T_2, \quad G = T_2 \cdot T_2,$$

We define:
$$W = EG - F^2$$

Instead of an orthonormal frame, it is more convenient to take a basis:

$$(4) \qquad T_1 = (1, 0, \varphi_x, \psi_x), \qquad\qquad T_2 = (0, 1, \varphi_y, \psi_y)$$
$$N_1 = (-\varphi_x, -\varphi_y, 1, 0), \qquad\qquad N_2 = (-\psi_x, -\psi_y, 0, 1)$$

The vectors T_1 and T_2 span the tangent space, and the vectors N_1 and N_2 span the normal space. We define:

$$\hat{E} = N_1 \cdot N_1, \quad \hat{F} = N_1 \cdot N_2, \quad \hat{G} = N_2 \cdot N_2,$$

and it is easy to verify that:

$$\hat{E}\hat{G} - \hat{F}^2 = W$$

Proposition 1. *The Gaussian curvature is given by:*

$$(5) \qquad\qquad K = \frac{1}{W^2}(\hat{E}H_\psi - \hat{F}Q + \hat{G}H_\varphi)$$

where:

$$H_f = Hess(f) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}, \quad Q = \begin{vmatrix} \varphi_{xx} & \varphi_{xy} \\ \psi_{xy} & \psi_{yy} \end{vmatrix} - \begin{vmatrix} \varphi_{xy} & \varphi_{yy} \\ \psi_{xx} & \psi_{xy} \end{vmatrix}$$

Proposition 2. *The normal curvature is given by:*

$$(6) \qquad\qquad \kappa = \frac{1}{W^2}(EL - FM + GN)$$

where:

$$L = \begin{vmatrix} \varphi_{xy} & \varphi_{yy} \\ \psi_{xy} & \psi_{yy} \end{vmatrix}, \quad M = \begin{vmatrix} \varphi_{xx} & \varphi_{yy} \\ \psi_{xx} & \psi_{yy} \end{vmatrix}, \quad N = \begin{vmatrix} \varphi_{xx} & \varphi_{xy} \\ \psi_{xx} & \psi_{xy} \end{vmatrix}$$

3. The Gauss map

For an immersed oriented two dimensional surface S in $\mathbf{R}^4$, the *Gauss map* takes a point p of the surface S to the oriented plane through the origin $[T_pS]$ which is parallel to the tangent plane T_pS to S at p. The image of this map is a subset of the Grassmannian manifold $G_{2,4}$, the set of oriented two planes in $\mathbf{R}^4$.

In $\mathbf{R}^4$ with oriented basis given by e_i, $i = 1, \ldots, 4$, we can completely determine an oriented plane P, defined by the ordered pair of perpendicular unit vectors $v_1 = c_1e_1 + c_2e_2 + c_3e_3 + c_4e_4$ and $v_2 = d_1e_1 + d_2e_2 + d_3e_3 + d_4e_4$, by giving the algebraic areas $p_{ij} = c_id_j - c_jd_i$ of the projections of the unit square associated to v_1, v_2 in that plane into the six coordinate planes $e_i \wedge e_j$, $i < j$:

$$P = p_{12}e_1 \wedge e_2 + p_{13}e_1 \wedge e_3 + p_{14}e_1 \wedge e_4 + p_{34}e_3 \wedge e_4 + p_{42}e_4 \wedge e_2 + p_{23}e_2 \wedge e_3$$

The six Plücker coordinates are $P = (p_{12}, p_{13}, p_{14}, p_{34}, p_{42}, p_{23})$, and could be obtained by taking the wedge product of any oriented pair of linearly independent vectors v_1 and v_2:

$$P = \frac{v_1 \wedge v_2}{|v_1 \wedge v_2|}$$

The set of two planes in four space is not six dimensional since there are two relations among these Plücker coordinates, as can be shown by direct calculation:

- $\sum_{i<j} p_{ij}^2 = 1$.
- $p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23} = 0$.

The first relation defines $\mathbf{S}^5 \subset \mathbf{R}^6$, and the two together the Klein quadric.

We define the Klein coordinates by:

$$(7) \qquad \begin{aligned} a_1 &= p_{12} + p_{34}, & a_2 &= p_{13} + p_{42}, & a_3 &= p_{14} + p_{23} \\ b_1 &= p_{12} - p_{34}, & b_2 &= p_{13} - p_{42}, & b_3 &= p_{14} - p_{23} \end{aligned}$$

The two algebraic conditions on the Plücker coordinates then become:

$$a_1^2 + a_2^2 + a_3^2 = 1, \quad b_1^2 + b_2^2 + b_3^2 = 1$$

Thus the vectors $a = (a_1, a_2, a_3)$ and $b = (b_1b_2, b_3)$ are unit vectors in three space, the sphere coordinates for a plane in $G_{2,4}$, and we can view $G_{2,4}$ as the product of two 2-spheres:

$$G_{2,4} \cong \mathbf{S}^2 \times \mathbf{S}^2$$

The Gauss map $p \mapsto [T_pS]$ will be denoted by:

$$\Gamma : S \longrightarrow G_{2,4}, \quad \Gamma = (\Gamma_1, \Gamma_2), \quad \Gamma_{1,2} : S \longrightarrow \mathbf{S}^2$$

We shall use Γ with the Plücker coordinates, and (Γ_1, Γ_2) with the Klein coordinates.

It is possible to define a symplectic form (or an area form) σ on $\mathbf{S}^2 \subset \mathbf{R}^3$ as follows:

$$(8) \qquad \sigma = i^*\zeta, \quad \zeta = x\mathrm{d}y \wedge \mathrm{d}z + y\mathrm{d}z \wedge \mathrm{d}x + z\mathrm{d}x \wedge \mathrm{d}y$$

where $i : \mathbf{S}^2 \longrightarrow \mathbf{R}^3$ is the standard inclusion.

If we consider a surface S locally given by a parametrization:

$$\Xi : (x, y) \mapsto (x, y, \varphi(x, y), \psi(x, y))$$

where the 1-jets of φ and ψ vanish at the origin, then:

$$T_1 \wedge T_2 = (1, \varphi_y, \psi_y, \varphi_x\psi_y - \varphi_y\psi_x, \psi_x, -\varphi_x), \quad |T_1 \wedge T_2| = \sqrt{W}$$

The Gauss map is given by:

$$(9) \qquad \Gamma_1 = \frac{1}{\sqrt{W}}(1 + \varphi_x\psi_y - \varphi_y\psi_x, -\varphi_x + \psi_y, -\varphi_y - \psi_x)$$

$$\Gamma_2 = \frac{1}{\sqrt{W}}(1 - \varphi_x\psi_y + \varphi_y\psi_x, -\varphi_x - \psi_y, -\varphi_y + \psi_x)$$

and the normal space is represented by $(\Gamma_1, -\Gamma_2)$.

There is a relation between the Gauss map and the Gaussian and normal curvature of the surface, as expressed by the following:

Theorem 1 (Blaschke [3]). *The pull-back of the form σ on $\mathbf{S}^2$ by a component of the Gauss map is:*

$$(10) \qquad \Gamma_1^*\sigma = (K + \kappa)\,\omega_1 \wedge \omega_2, \quad \Gamma_2^*\sigma = (K - \kappa)\,\omega_1 \wedge \omega_2$$

Remark 1. The above statement is equivalent to:

$$\det \mathrm{D}\Gamma_1 = K + \kappa, \quad \det \mathrm{D}\Gamma_2 = K - \kappa$$

The Grassmannian $G_{2,4}$ can also be considered a symplectic manifold in two natural ways, with symplectic forms $\Omega_\pm$ defined as follows:

$$\Omega_+ = \frac{1}{2}\left(\pi_a^*\sigma_a + \pi_b^*\sigma_b\right), \quad \Omega_- = \frac{1}{2}\left(\pi_a^*\sigma_a - \pi_b^*\sigma_b\right)$$

Corollary 1. *The pull-back of the symplectic forms $\Omega_\pm$ on $G_{2,4}$ by the Gauss map is:*

$$(11) \qquad \Gamma^*\Omega_+ = K\,\omega_1 \wedge \omega_2, \quad \Gamma^*\Omega_- = \kappa\,\omega_1 \wedge \omega_2$$

4. ISOCLINIC SURFACES

Wong [12] developed a curvature theory for surfaces in $\mathbf{R}^4$ based on the two angles between two tangent planes of the surface. We are interested in conditions under which those two angles be equal:

Definition 1. Two planes in $\mathbf{R}^4$, passing through the origin, are isoclinic if the the unit circumference centered at the origin in one of them projects as a circumference in the other one.

Definition 2. The isocline surface $\mathcal{I}_P$ of a plane P in $\mathbf{R}^4$ is the subset of $G_{2,4}$ consisting of the planes P' that are isoclinic to P.

We take the (x, y)-plane, defined by $u = 0$, $v = 0$. A plane P defined by:

$$u = \alpha_1 x + \beta_1 y, \quad v = \alpha_2 x + \beta_2 y$$

is isoclinic to the (x, y)-plane if the equation:

$$x^2 + y^2 + (\alpha_1 x + \beta_1 y)^2 + (\alpha_2 x + \beta_2 y)^2 = 1$$

defines a circumference in the (x, y)-plane. This is equivalent to:

$$(12) \qquad \qquad \|\alpha\|^2 = \|\beta\|^2, \quad \alpha \cdot \beta = 0$$

where $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$. The equations (12) for (α, β) define a singular surface $\mathcal{I}$ in $\mathbf{R}^4$; the unique singularity is at the origin $(0, 0, 0, 0)$, corresponding to the (x, y)-plane.

We can view $\mathcal{I}$ as the union $\mathcal{I} = \mathcal{I}^+ \cup \mathcal{I}^-$ of two orthogonal planes in $\mathbf{R}^4$:

$$\mathcal{I}^\pm : \qquad \beta_1 = \mp\alpha_2, \quad \beta_2 = \pm\alpha_1$$

Let C be the linear map exchanging the third and fourth coordinates:

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Then:

$$\mathcal{I}^+ = C\mathcal{I}^-$$

The above is a parametrization of a part of the isocline surface. In fact, the isocline surface $\mathcal{I}$ of the (x, y) plane is a compact subset of $G_{2,4}$, the union of the two sets:

$$\mathcal{I}^+ = \{P \in G_{2,4} \mid a(P) = e_1 \in S^2\}, \quad \mathcal{I}^- = \{P \in G_{2,4} \mid b(P) = e_1 \in S^2\}$$

where $(a(P), b(P))$ denotes the sphere coordinates on $G_{2,4}$.

Proposition 3. *The isocline surface $\mathcal{I}_P$ of the plane P is the union of the orbit of P under the action of $SU(2)$ in $\mathbf{R}^4$ and of its image under the linear map C :*

$$\mathcal{I}_P = \{AP \mid A \in SU(2)\} \cup \{CAP \mid A \in SU(2)\}$$

Definition 3. Let $u \in T_pS$ with $\|u\| = 1$; then u spans an isoclinic direction of type $+$ $(-)$ if there exists a curve $\gamma(t)$ on S such that:

$$\gamma(0) = p, \quad \dot{\gamma}(0) = u, \qquad \dot{P}(0) \in T_P\mathcal{I}_P^+ \ (T_P\mathcal{I}_P^-)$$

with $P(t) = T_{\gamma(t)}S$, $P = P(0) = T_pS$.

Theorem 2 (Wong [12]). *There exists an isoclinic direction at a given point $p \in S$ if, and only if:*

(13) $$|K(p)| = |\kappa(p)|$$

Proposition 4. *The isoclinic direction is defined by:*

$$(a \pm f)\omega_1 + (b \pm g)\omega_2 = 0, \quad as \ K = \pm\kappa$$

Definition 4. The surface S is isoclinic , or a I-isurface, f there exists (at least) an isoclinic direction at every point of S.

We can consider a line field $\mathfrak{I}$ on any isoclinic surface, taking the isoclinic direction at every point, and also a particular vector field $X_{\mathcal{I}}$ spanning it, given by:

$$X_{\mathcal{I}} \rfloor \omega_1 \wedge \omega_2 = (a \pm f)\,\omega_1 + (b \pm g)\,\omega_2$$

We consider the $+$ case, when $K = \kappa$; the other one is similar.

Theorem 3. *The vector field $X_{\mathcal{I}}$ is symplectic.*

Proof. We have to prove that $(a + f)\,\omega_1 + (b + g)\,\omega_2$ is closed. Using the Monge normal form around an arbitrary $p \in S$ we see that:

$$(a + f)\,\omega_1 + (b + g)\,\omega_2 = (\varphi_{xx} + \psi_{xy})\mathrm{d}x + (\varphi_{xy} + \psi_{yy})\mathrm{d}y + O(2)$$

and as the exterior derivative at a point only depends on the coefficients and its derivatives at that point:

$$\mathrm{d}\left[(a + f)\,\omega_1 + (b + g)\,\omega_2\right] = \mathrm{d}\left[(\varphi_{xx} + \psi_{xy})\mathrm{d}x + (\varphi_{xy} + \psi_{yy})\mathrm{d}y\right] = 0$$

at the origin, corresponding to $p \in S$. $\qquad\qquad\square$

5. LAGRANGEAN SURFACES

A *symplectic manifold* is a pair (M, ω), where M is a $2n-$dimensional differentiable manifold and ω is a symplectic form: a closed non degenerate 2-form. Then:

$$\Omega = \frac{1}{n!}\, \omega^n \ \text{ is a volume form, and } \mathrm{d}\omega = 0$$

A *symplectic map* is a map $\varphi : (M, \omega) \longrightarrow (M', \omega')$, such that:

$$\varphi^* \omega' = \omega$$

A *Lagrangean submanifold* L of (M, ω) is an immersed submanifold of M such that:

$$i^* \omega \equiv 0, \quad \text{where } i : L \longrightarrow M \text{ is the immersion map}$$

Definition 5. A *Lagrangean surface* $\mathcal{L}$ is an immersed Lagrangean submanifold of $(\mathbf{R}^4, \omega)$.

Local Normal Form. *Given $p \in \mathcal{L}$, there is a change of coordinates, by a translation and a linear symplectic and orthogonal map, such that locally $\mathcal{L}$ becomes the graph around the origin of a map*

$$\Phi = (\varphi, \psi) : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$$

satisfying:

- *The first jet of Φ is zero at the origin.*
- $\dfrac{\partial \varphi}{\partial y} \equiv \dfrac{\partial \psi}{\partial x}$

Remark 2. If we preserve orientation, so that the linear map $A \in SO(4)$, there is another normal form; the symplectic form in $\mathbf{R}^4$ is then $\omega' = \mathrm{d}x \wedge \mathrm{d}u - \mathrm{d}y \wedge \mathrm{d}v$ and the identity in the normal form is:

$$\frac{\partial \varphi}{\partial y} \equiv -\frac{\partial \psi}{\partial x}$$

Proposition 5. *A necessary condition for $\mathcal{L} \subset \mathbf{R}^4$ to be a Lagrangean surface is that the Gaussian curvature and the normal curvature coincide up to sign:*

$$|K| \equiv |\kappa|$$

Proof. Assume that $\mathcal{L} \subset \mathbf{R}^4$ is Lagrangean, and locally given by a parametrization as in the normal form:

$$(x, y) \mapsto (x, y, \varphi(x, y), \psi(x, y)), \quad \varphi_y \equiv \psi_x$$

where $\Phi = (\varphi, \psi)$ has vanishing first jet at the origin, $j^1\Phi(0) = 0$.

It follows from $\varphi_y \equiv \psi_x$ that in the moving frame associated to the local normal form we have:

(14) $\hat{E} = E, \quad \hat{F} = F, \quad \hat{G} = G, \qquad L = H_\psi, \quad M = Q, \quad N = H_\varphi$

and therefore:

$$K \equiv \kappa$$

If instead we have $\varphi_y \equiv -\psi_x$, then in the moving frame associated to the local normal form:

$$\hat{E} = E, \quad \hat{F} = -F, \quad \hat{G} = G, \qquad L = -H_\varphi, \quad M = Q, \quad N = -H_\psi$$

and so:

$$K \equiv -\kappa$$

$\square$

Remark 3. If $\mathcal{L} \subset \mathbf{R}^4$ is a Lagrangean surface, in the moving frame associated to the local normal form we also have:

(15) $e \equiv b, \quad f \equiv c$

The condition $|K| \equiv |\kappa|$ is not a sufficient condition for the surface to be Lagrangean, as there are rigid motions that do not preserve the symplectic form while naturally preserving the Gauss curvature and the normal curvature. The example 2 shows that $|K| \equiv |\kappa|$ is not a sufficient condition for the surface to be congruent to a Lagrangean surface.

Corollary 2. *A necessary condition for $\mathcal{L} \subset \mathbf{R}^4$ to be a Lagrangean surface is that it be an isoclinic surface: there exists an isoclinic direction at every point of $\mathcal{L}$.*

5.1. **Holomorphic graphs and R-surfaces.** Consider the holomorphic map:

$$f : U \subset \mathbf{C} \longrightarrow \mathbf{C}$$

Then $\alpha(z) = (z, f(z))$ defines a plane complex curve. We can determine its curvature by:

$$k(z) = \frac{\alpha'(z) \wedge \alpha''(z)}{|\alpha'(z)|^3} = \frac{1}{|\alpha'(z)|^3} f''(z)$$

and its inflection points, where the second derivative $f''(z)$ vanishes, are also the points where the curvature vanishes.

If we consider:

$$S = \alpha(U) = \text{graph } f \subset \mathbf{C}^2 \cong \mathbf{R}^4$$

we obtain a surface in $\mathbf{R}^4$. Taking $z = x + iy$ and $f(z) = \varphi(x, y) + i\psi(x, y)$, a parametrization is given by

$$\Xi(x, y) = (x, y, \varphi(x, y), \psi(x, y))$$

and then the Gaussian curvature is:

$$K = -\frac{2}{(1 + \varphi_x^2 + \psi_x^2)^3}(\varphi_{xx}^2 + \psi_{xx}^2) \leq 0$$

the mean curvature:

$$\mathcal{H} \equiv 0$$

and the normal curvature:

$$\kappa = \frac{2}{(1 + \varphi_x^2 + \psi_x^2)^3}(\varphi_{xx}^2 + \psi_{xx}^2) = -K$$

The surface S is a Lagrangean surface with respect to two distinct symplectic forms in $\mathbf{R}^4$:

$$\Omega_1 = \mathrm{d}x \wedge \mathrm{d}u - \mathrm{d}y \wedge \mathrm{d}v, \quad \Omega_2 = \mathrm{d}x \wedge \mathrm{d}v + \mathrm{d}y \wedge \mathrm{d}u$$

since:

$$\Xi^*\Omega_1 = \varphi_y \mathrm{d}x \wedge \mathrm{d}y - \psi_x \mathrm{d}y \wedge \mathrm{d}x \equiv 0$$

and

$$\Xi^*\Omega_2 = \psi_y \mathrm{d}x \wedge \mathrm{d}y + \varphi_x \mathrm{d}y \wedge \mathrm{d}x \equiv 0$$

The tangent vectors:

$$T_1 = (1, 0, \varphi_x, \psi_x), \quad T_2 = (0, 1, \varphi_y, \psi_y)$$

are orthogonal and have the same norm, from the Cauchy-Riemann equations. The Gauss map $(x, y) \mapsto e_1 \wedge e_2$ is given by:

$$e_1 \wedge e_2 = \frac{1}{1 + \varphi_x^2 + \psi_x^2}(1, -\psi_x, \varphi_x, \varphi_x^2 + \psi_x^2, \psi_x, -\varphi_x)$$

or, in sphere coordinates (a, b), by:

$$a = \frac{1}{1 + \varphi_x^2 + \psi_x^2}(1 + \varphi_x^2 + \psi_x^2, \varphi_x - \varphi_x, \psi_x - \psi_x) = (1, 0, 0)$$

$$b = \frac{1}{1 + \varphi_x^2 + \psi_x^2}(1 - \varphi_x^2 - \psi_x^2, \varphi_x + \varphi_x, \psi_x + \psi_x) =$$

$$= \frac{1}{1 + \varphi_x^2 + \psi_x^2}(1 - \varphi_x^2 - \psi_x^2, 2\varphi_x, 2\psi_x)$$

Definition 6. S is a R-surface if it is locally congruent to the real version of a holomorphic graph.

Thus we see that for R-surfaces:

- All tangent planes are isoclinic to the (x, y) plane, so they all belong to $\mathcal{I}$, in fact to $\mathcal{I}^+$.

- The image of one component of the Gauss map is a point, $\Gamma_1 \equiv (1, 0, 0)$.
- All directions are isoclinic.

A straightforward computation shows that the singularities of the Gauss map, where $(x, y) \mapsto b(x, y)$ fails to be an immersion, are given by the inflection points, where the Gauss curvature vanishes: $\varphi_{xx} = \psi_{xx} = 0$.

6. Inflection points

The points p where the curvature ellipse passes through the origin are characterised by $\Delta(p) = 0$, where:

(16)
$$\Delta = \frac{1}{4} \begin{vmatrix} a & 2b & c & 0 \\ e & 2f & g & 0 \\ 0 & a & 2b & c \\ 0 & e & 2f & g \end{vmatrix}$$

In fact, Δ is the resultant of the two polynomials $\mathbf{II}_1 = ax^2 + 2bxy + cy^2$ and $\mathbf{II}_2 = ex^2 + 2fxy + gy^2$. If there exists u such that $\mathbf{II}(u) = 0$ those polynomials have a common root, and their resultant has to be zero.

A straightforward computation gives:

(17)
$$\Delta = (ac - b^2)(eg - f^2) - \frac{1}{4}(ag + ce - 2bf)^2 =$$
$$= (af - be)(bg - cf) - \frac{1}{4}(ag - ce)^2 = \det \mathfrak{N}$$

where:

$$\mathfrak{N}(x, y) = (af - be)x^2 + (ag - ce)xy + (bg - cf)y^2$$

and therefore the equation $\mathfrak{N}(x, y) = 0$ defining the asymptotic directions [8], given by (x, y), has two solutions, one or no solutions as $\Delta(p) < 0$, $\Delta(p) = 0$ or $\Delta(p) > 0$, respectively.

The points of S may be classified [8] using Δ, as follows:

Definition 7. A pont $p \in S$ is hyperbolic, parabolic or elliptic as $\Delta(p) < 0$, $\Delta(p) = 0$ or $\Delta(p) > 0$, respectively.

When $\Delta(p) = 0$ and the normal curvature vanishes (the curvature ellipse is contained in a line segment), $\kappa(p) = 0$, we can distinguish among the following special points:

- $\Delta(p) = 0$, $K(p) < 0$
  p is an *inflection point of real type*: the curvature ellipse is a radial segment and p does not belong to it.

- $\Delta(p) = 0$, $K(p) = 0$
  p is an *inflection point of flat type*: the curvature ellipse is a radial segment and p belongs to its boundary.
- $\Delta(p) = 0$, $K(p) > 0$
  p is an *inflection point of imaginary type*: the curvature ellipse is a radial segment and p belongs to its interior.

We have seen before (14) that on a Lagrangean surface we have:

$$L = H_\psi \quad M = Q, \quad N = H_\varphi$$

and also (15):

$$e \equiv b, \quad f \equiv c \text{ and thus } af - be = ac - b^2, \quad bg - cf = eg - f^2$$

Proposition 6. *Let S be an isoclinic surface and $p \in S$. Then the following conditions are equivalent:*

(1) *p is an inflection point.*
(2) *p is a parabolic point, $\Delta(p) = 0$, where the normal curvature vanishes, $\kappa(p) = 0$.*
(3) *rank $\begin{bmatrix} a & b & c \\ e & f & g \end{bmatrix} \leq 1$.*
(4) *p is a flat inflection point.*
(5) *p is a parabolic point where the Gaussian curvature vanishes, $K(p) = 0$.*
(6) *rank $\begin{bmatrix} a & b & e & f \\ b & c & f & g \end{bmatrix} \leq 1$.*
(7) *The Gauss map $\Gamma : S \longrightarrow G_{2,4} \subset \mathbf{S}^5$ is not an immersion at p.*

Proof. It is proved in [8] that the first three statements are equivalent, and the last four are also equivalent and imply the former; but, since $|K| = |\kappa|$, the third and fifth statements are equivalent, therefore they all are equivalent. $\qquad\square$

Proposition 7. *If the Gaussian curvature vanishes at a point p in the discriminant curve $\Delta = 0$, then the discriminant curve has a non Morse singularity at p.*

Proof. The Gauss curvature of a surface in $\mathbf{R}^4$ is the sum of the Gaussian curvatures of its projections in $\mathbf{R}^3$ along two orthogonal normal directions, that we can take as being e_3 and e_4:

(18) $$K = K_1 + K_2, \quad K_1 = ac - b^2, \quad K_2 = eg - f^2$$

and if $K(p) = 0$ we have:

$$K_1(p) = -K_2(p)$$

The equation $\Delta = 0$ at the same point gives:

(19) $K_1(p) = 0, \quad K_2(p) = 0, \quad (ag + ce - 2bf)(p) = 0$

and therefore p is a singular point:

$$\Delta_x(p) = 0, \quad \Delta_y(p) = 0$$

The singularity is non Morse if the Hessian vanishes, $H_\Delta(p) = 0$; it is proved [9, 4] that there exists a positive real number ξ such that, at a critical point, $H_\Delta(p) = \xi K(p)$.   □

Proposition 8. *If a point in surface S is a flat inflection point then, for any projection on $\mathbf{R}^3$ along a normal direction, its image belongs to the parabolic curve of that projection, where the Gaussian curvature vanishes.*

Proof. A point is a flat inflection point if it belongs to the intersection of the discriminant curve $\Delta = 0$ and the curve of zero Gaussian curvature $K = 0$; we have already seen (19) that $K_1 = K_2 = 0$ at any point where $\Delta = 0$ and $K = 0$.

If $K_1 = K_2 = 0$ the projection of the point belongs to the parabolic curve of the surface in $\mathbf{R}^3$ obtained by projections along the normal directions given by e_3 and e_4. It is easy to see that the same is true if the direction is given by a linear combination of e_3 and e_4.   □

Also, if the image belongs to the parabolic curve for any projection on $\mathbf{R}^3$ along a normal direction, this is in particular true for the directions given by e_3 and e_4 and $K_1 = K_2 = 0$.

Our interest in the inflection points in a Lagrangean surface is the study of the Gauss map around them, therefore we have to make changes of coordinates that are simultaneously isometries and symplectomorphisms, as they do not affect the metric properties nor the symplectic (or Lagrangean) properties.

Lemma 1. *Let $p \in S$ be a point in a Lagrangean surface, given in normal form as the graph around the origin of a map $(x, y) \mapsto (\varphi(x, y), \psi(x, y))$. If $K_1(p) = 0$ and $K_2(p) = 0$, then p is an inflection point, and:*

$$j^3\varphi = \frac{1}{2}Cx^2 + \frac{1}{3}ax^3 + bx^2y + cxy^2 + \frac{1}{3}dy^3, \quad j^3\psi = \frac{1}{3}bx^3 + cx^2y + dxy^2 + e\frac{1}{3}y^3$$

Proof. At $H_\varphi(0,0) = H_\psi(0,0) = 0$ we see that S is the graph of $(x, y) \mapsto (\varphi(x, y), \psi(x, y))$, with:

$$j^2\varphi = \frac{1}{2}A(a_1x + a_2y)^2, \qquad j^2\psi = \frac{1}{2}B(b_1x + b_2y)^2$$

and $\|a\| = \|b\| = 1$, where $a = (a_1, a_2)$, $b = (b_1, b_2)$.

From $K(0,0) = H_\varphi(0,0) + H_\psi(0,0) = 0$ it follows that $0 = \kappa(0,0) = L(0,0) + N(0,0)$ and we also have:

$$\begin{vmatrix} \varphi_{xy} & \psi_{xy} \\ \varphi_{yy} & \psi_{yy} \end{vmatrix} + \begin{vmatrix} \varphi_{xx} & \psi_{xx} \\ \varphi_{xy} & \psi_{xy} \end{vmatrix} = 0$$

at the origin. This is readily evaluated as:

$$AB(a \cdot b)(a \wedge b) = 0$$

In the generic case $AB \neq 0$ and thus either $a = \pm b$ or $a \perp b$; if we assume $a \cdot b = 0$, or $b_1 = a_2$, $b_2 = -a_1$, the conditions:

$$\varphi_{xy}(0,0) = \psi_{xx}(0,0), \quad \varphi_{yy}(0,0) = \psi_{xy}(0,0)$$

are equivalent to:

$$A a_1 a_2 = B a_2^2, \quad A a_2^2 = -B a_1 a_2$$

These are impossible if we assume the generic condition $\varphi_{xy}(0,0) \neq 0$; under this condition we have $a \wedge b$, or $a = \pm b$ and also:

$$B = A \frac{a_2}{a_1}$$

so that:

$$j^2 \varphi = \frac{1}{2} A (a_1 x + a_2 y)^2, \qquad j^2 \psi = \frac{1}{2} A \frac{a_2}{a_1} (a_1 x + a_2 y)^2$$

The change of coordinates:

$$\xi = a_1 x + a_2 y \quad \eta = a_2 x - a_1 y$$
$$\zeta = a_1 u + a_2 v \quad w = a_2 u - a_1 v$$

is a symplectomorphism and an isometry, so it belongs to $SU(2)$ and takes isoclinic planes into isoclinic planes. In these coordinates (but writing with the old variables) we get:

$$j^2 \varphi = \frac{1}{2} C x^2, \quad j^2 \psi = 0, \qquad C = A \left(a_1 + \frac{a_2^2}{a_1} \right)$$

$\square$

Remark 4. If the surface is isoclinic, it is still true that:

$$j^2 \varphi = \frac{1}{2} C x^2, \quad j^2 \psi = 0, \qquad C = A \left(a_1 + \frac{a_2^2}{a_1} \right)$$

In this case, the situation $a \cdot b = 0$ is impossible, because at an inflection point there exists a linear combination of φ and ψ with vanishing zero 2-jet [9].

Theorem 4. *The existence of a flat inflection point in a generic La-grangean surface is a stable situation inside the class of Lagrangean surfaces.*

Proof. All inflection points p of a Lagrangean surface $\mathcal{L}$ are flat inflection points, and we have $\Delta(p) = 0$, $K(p) = 0$, and even $K_1(p) = K_2(p) = 0$.

Assume now that $K_1(p) = K_2(p) = 0$; then using the normal form around p of lemma 1, we get $\Delta = 0$ and $K = K_1 + K_2 = 0$ at p. This shows that p is a flat inflection point.

Locally, a Lagrangean surface $\mathcal{L}$ is given by the graph around the origin of a map $(x, y) \mapsto (\varphi(x, y), \psi(x, y))$, for which $\varphi_y \equiv \psi_x$. So we can identify $\mathcal{L}$ with a map F such that:

$$\varphi = F_x, \quad \psi = F_y$$

The conditions $K_1 = 0$, or $K_2 = 0$, are represented in the 3-jet space $J^3(\mathbf{R}^2, \mathbf{R})$ as closed algebraic sets of codimension one, and $K_1 = K_2 = 0$ as a closed algebraic set of codimension two. For a residual set of maps F, j^3F intersects transversally those stratified sets, therefore $K_1 = 0$ and $K_2 = 0$ are smooth curves intersecting transversally at isolated points corresponding to $K_1 = K_2 = 0$.

Assume that for a given F, or equivalently, for a given Lagrangean surface $\mathcal{L}$ that intersection is non empty: the curves $K_1 = 0$ and $K_2 = 0$ are smooth curves intersecting transversally at p, or j^3F intersects transversally the algebraic sets representing $K_1 = 0$, $K_2 = 0$ and $K_1 = K_2 = 0$; then near p so does the 3-jet of any small perturbation of F. $\qquad \square$

6.1. **Singularities of the Gauss map.** A generic map from $\mathbf{R}^2$ into $\mathbf{R}^4$ has no stable singularities. This is true also fo the Gauss map of a generic surface in $\mathbf{R}^4$, since in general the intersection of $\Delta = 0$ with $K(p) = 0$ is empty [8].

The situation is quite different for the Gauss map of Lagrangean and isoclinic surfaces: then the Gauss map is locally a map from $\mathbf{R}^2$ into $\mathbf{R}^3$ and it can present cross-cap singularities in a stable way [10].

Proposition 9. *Let S be an isoclinic surface. Then:*
- *the singularities of its Gauss map are inflection points;*
- *for a generic isoclinic or Lagrangean surface, the inflection points are cross-cap singularities of the Gauss map.*

Proof. We assume S in Monge form:

$$(x, y) \mapsto (x, y, \varphi(x, y), \psi(x, y))$$

with the point of interest being the image of the origin, and the tangent plane there being the (x, y)-plane, $z = 0, w = 0$.

We have already seen that the singularities of the Gauss map are flat inflection points of the surface. Assume now that $H_\varphi(0,0) = H_\psi(0,0) = 0$.

A parametrization of the Grassmannian $G_{2,4}$ is given by $z = \alpha_1\xi + \beta_1\eta$ and $w = \alpha_2\xi + \beta_2(\eta$, and the Gauss map can be written as:

$$\alpha(x, y) = (\varphi_x(x, y), \psi_x(x, y)), \quad \beta(x, y) = (\varphi_y(x, y), \psi_y(x, y))$$

The Gauss map is singular when the matrix:

$$\begin{bmatrix} \varphi_{xx} & \psi_{xx} & \varphi_{xy} & \psi_{xy} \\ \varphi_{yx} & \psi_{yx} & \varphi_{yy} & \psi_{yy} \end{bmatrix}$$

has rank one (or zero) at the origin. In the generic case we can use the normal form of lemma 1, or the subsequent remark, and then the matrix above becomes:

$$\begin{bmatrix} C & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The Gauss map fails to be an immersion, and therefore it has a singularity. For a generic Lagrangean or isoclinic surface $C \neq 0$, so the singularity is a cross-cap. $\qquad \square$

7. The Gauss map for Lagrangean surfaces

Theorem 5. *A necessary and sufficient condition for a surface $S \subset \mathbf{R}^4$ with Gauss map $\Gamma : S \longrightarrow G_{2,4} \subset \mathbf{S}^5$ to be congruent to a Lagrangean surface is that the image of Γ_1, or Γ_2, be contained in a great circle.*

Proof. Assume the immersion:

$$\Xi : U \subset \mathbf{R}^2 \longrightarrow \mathbf{R}^4$$

defines a Lagrangean surface of $\mathbf{R}^4$ with symplectic form $\omega = \mathrm{d}x \wedge \mathrm{d}u + \mathrm{d}y \wedge \mathrm{d}v$. Then $\Xi^*\omega \equiv 0$ or equivalently:

$$p_{13} + p_{24} \equiv 0$$

Recalling the definition of Γ, we see that its image is contained in $\mathbf{S}^5 \cap \{p_{13} - p_{42} = 0\}$, a great circle. In sphere coordinates this is equivalent to $b_2 = 0$, thus he image of Γ_2 is contained in a great circle. The other case $a_2 = 0$) corresponds to the opposite orientation.

Assume instead that the immersion Ξ has a Gauss map such that the image of Γ_2 is contained in a great circle. This means there exists a vector $\alpha \in \mathbf{S}^2 \subset \mathbf{R}^3$ such that $\alpha \cdot b = 0$ defines the plane passing through the origin containing the image of Γ_2. We have to prove that

there exists a rigid motion in $\mathbf{R}^4$, taking S into $\hat{S}$, such that the image of $\hat{\Gamma}_2$ is contained in $b_2 = 0$.

It is clear that translations in $\mathbf{R}^4$ do not affect the Gauss map. The linear action of $\mathrm{SO}(4)$ on $\mathbf{R}^4$ induces an action on $G_{2,4}$, that can be extended to $\mathbf{R}^6$ as a linear action:

Lemma 2. *If $A \in \mathrm{SO}(4)$ there exists a linear operator $\mathfrak{A}$ on $\mathbf{R}^6$ such that if $P = v_1 \wedge v_2$ then:*

$$Av_1 \wedge Av_2 = \mathfrak{A}P$$

Proof. Let A^i be the ith-column of A in the standard basis, identified with the vector $\sum A^i_k e_k$, and let $\mathfrak{A}^{ij} = A^i \wedge A^j$ be a column of $\mathfrak{A}$ (again the vector $\sum A^{ij}_{kl} e_k \wedge e_l$ is the exterior product of the vectors represented by A^i and A^j):

$$A = [A^1 : A^2 : A^3 : A^4], \quad \mathfrak{A} = [\mathfrak{A}^{12} : \mathfrak{A}^{13} : \mathfrak{A}^{14} : \mathfrak{A}^{34} : \mathfrak{A}^{42} : \mathfrak{A}^{23}]$$

A simple computation shows that $\mathfrak{A}$ is the desired linear operator on $\mathbf{R}^6$; a similar construction can be made based on the lines of the two matrices. $\square$

Lemma 3. *The action of $\mathrm{SO}(4)$ on $\mathbf{R}^6$ is orthogonal:*

$$(A, P) \mapsto \mathfrak{A}P, \quad \mathfrak{A} \in \mathrm{O}(6)$$

Proof. The action of $\mathrm{SO}(4)$ on $\mathbf{R}^6$ is defined by: $(A, P) \mapsto \mathfrak{A}P$. This is indeed a well defined action since:

$$C = AB \implies \mathfrak{C} = \mathfrak{A}\mathfrak{B}$$

where $A, B, C \in \mathrm{SO}(4)$ and $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ are the corresponding linear operators on $\mathbf{R}^6$. Then $\mathfrak{A}\mathfrak{A}^T = I$ follows from $AA^T = I$. $\square$

Lemma 4. $\mathbf{S}^5$ *and $G_{2,4}$ are invariant for the action induced by $\mathrm{SO}(4)$:*

$$\mathfrak{A}\mathbf{S}^5 \subset \mathbf{S}^5, \quad \mathfrak{A}\mathbf{G}_{2,4} \subset \mathbf{G}_{2,4}, \quad \text{for all } A \in \mathrm{SO}(4)$$

Moreover the action is transitive on $G_{2,4}$.

Proof. The invariance of $\mathbf{S}^5$ follows from the action being orthogonal, and then the invariance of $G_{2,4}$ is a consequence of the action of $\mathrm{SO}(4)$ on $\mathbf{R}^4$ take planes into planes.

Let P_1 and P_2 be any two points in $G_{2,4}$, and $\{u_i\}$, $\{v_i\}$, $i = 1, \ldots, 4$ be orthonormal bases of $\mathbf{R}^4$ with the same orientation such that $u_1, u_2 \in P_1$ and $v_1, v_2 \in P_2$; there exists $A \in \mathrm{SO}(4)$ taking one basis into the other, therefore taking the plane P_1 into the plane P_2 (note they do not need to have the same orientation). Then $\mathfrak{A}P_1 = P_2$. $\square$

Now $\alpha \cdot b = 0$ defines the plane passing through the origin containing the image of Γ_2; this means that the image of (Γ_1, Γ_2) is contained in the hyperplane $(0, \alpha) \cdot (a, b) = 0$, or that the image of Γ is contained in the hyperplane $(\alpha, \alpha) \cdot P = 0$.

If there exists $A \in \mathrm{SO}(4)$ such that:

$$\mathfrak{A}(\alpha, \alpha) = \beta, \quad \beta = (0, 1, 0, 0, -1, 0)$$

then, for all P in the plane $(\alpha, \alpha) \cdot P = 0$:

$$\beta \cdot (\mathfrak{A}P) = (\mathfrak{A}(\alpha, \alpha)) \cdot (\mathfrak{A}P) = (\alpha, \alpha) \cdot P = 0$$

Therefore $\mathfrak{A}$ takes the plane $(\alpha, \alpha) \cdot P = 0$ into the plane $p_{13} - p_{42} = p_{13} + p_{24} = 0$, and the surface $A\Xi$ is Lagrangean.

Consider $\hat{A} \in \mathrm{SO}(4)$ defined by:

$$\hat{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{\alpha_2}{\sqrt{\alpha_1^2 + \alpha_2^2}} & \alpha_1 & \dfrac{\alpha_1 \alpha_3}{\sqrt{\alpha_1^2 + \alpha_2^2}} \\ 0 & -\dfrac{\alpha_1}{\sqrt{\alpha_1^2 + \alpha_2^2}} & \alpha_2 & \dfrac{\alpha_2 \alpha_3}{\sqrt{\alpha_1^2 + \alpha_2^2}} \\ 0 & 0 & \alpha_3 & -\dfrac{(\alpha_1^2 + \alpha_2^2)}{\sqrt{\alpha_1^2 + \alpha_2^2}} \end{bmatrix}$$

Then:

$$\hat{\mathfrak{A}}(0, 1, 0, 0, -1, 0) = (\alpha, \alpha)$$

and $A = \hat{A}^{-1}$ gives the required rigid motion. $\qquad \square$

Remark 5. These lemmas can be extended to arbitrary n and k, the action of $\mathrm{SO}(n)$ on $\mathbf{R}^n$ inducing an orthogonal action on the Euclidean space $\mathbf{R}^N$ containing $G_{k,n}$ with the same properties.

Example 1. Consider the surface S given by:

$$\varphi(x, y) = x^2 - y^2, \quad \psi(x, y) = ax + by - 2xy$$

A straightforward computation shows that:

$$H_\varphi = H_\psi = L = N = -4, \quad M = Q = 0$$

therefore:

$$K = \kappa \Longleftrightarrow E + G = \hat{E} + \hat{G}$$

This is easily verified, as:

$$E = 1 + 4x^2 + (a - 2y)^2, \qquad \hat{E} = 1 + 4x^2 + 4y^2$$
$$G = 1 + 4y^2 + (b - 2x)^2, \qquad \hat{G} = 1 + (a - 2y)^2 + (b - 2x)^2$$

The surface has Gauss curvature and normal curvature that coincide, but if a is not zero we have $\psi_x \not\equiv \varphi_y$, and if b is not zero we have $\psi_y \not\equiv -\varphi_x$.

On the other hand, the image of Γ_2 is the curve:

$$\frac{1}{\sqrt{W}}(\sqrt{W - a^2 - b^2}, b, -a)$$

which is contained in the great circle $\mathbf{S}^2 \cap \{ab_2 + bb_3 = 0\}$; the image of Γ is contained in the plane:

$$\alpha \cdot P = 0, \quad \alpha = \frac{1}{\sqrt{2(a^2 + b^2)}}(0, -a, b, 0, a, -b)$$

and is therefore contained in a great circle.

Then there exists a linear map $A \in \mathrm{SO}(4)$ in $\mathbf{R}^4$ (lemma 4) that takes the above surface to a Lagrangean surface, in fact:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -b/\sqrt{a^2 + b^2} & -a/\sqrt{a^2 + b^2} \\ 0 & 0 & a/\sqrt{a^2 + b^2} & -b/\sqrt{a^2 + b^2} \end{bmatrix}$$

The image $\hat{S}$ of S by A is again parametrized in the form:

$$(x, y) \mapsto (x, y, \hat{\varphi}(x, y), \hat{\psi}(x, y))$$

with

$$\hat{\varphi}(x, y) = \frac{1}{\sqrt{a^2 + b^2}}\left(a(2xy - ax - by) - b(x^2 - y^2)\right)$$

$$\hat{\psi}(x, y) = \frac{1}{\sqrt{a^2 + b^2}}\left(b(2xy - ax - by) + a(x^2 - y^2)\right)$$

Then:

$$\frac{\partial \hat{\varphi}}{\partial y}(x, y) \equiv \frac{\partial \hat{\psi}}{\partial x}(x, y)$$

and therefore the surface $\hat{S}$ is Lagrangean.

Example 2. Consider a surface S such the image of Γ_2 is the curve:

$$b_1 = c, \quad 0 < |c| < 1$$

which is not contained in any great circle. This condition above is equivalent to:

$$1 + \phi_x \psi_y - \phi_y \psi_x = c\sqrt{1 + \phi_x^2 + \phi_y^2 + \psi_x^2 + \psi_y^2 + (\phi_x \psi_y - \phi_y \psi_x)^2}$$

If we fix:

$$\psi(x, y) = \frac{1}{2}(x^2 + y^2)$$

we obtain a first order partial differential equation for ϕ:

$$1 - y\phi_x + x\phi_y = c\sqrt{1 + \phi_x^2 + \phi_y^2 + x^2 + y^2 + (y\phi_x - x\phi_y)^2}$$

We consider the initial condition: $\phi(x, 0) = -x^2/2$, and therefore $\phi_x(x, 0) = -x$; then $\phi_y(x, 0)$ has to satisfy:

$$1 + x\phi_y(x, 0) = c\sqrt{1 + 2x^2 + \phi_y(x, 0)^2 + x^2\phi_y(x, 0)^2}$$

If we choose $c = 1/\sqrt{2}$ then at the origin:

$$2 = 1 + \phi_y(0, 0)^2, \quad \text{or } \phi_y(0, 0) = \pm 1$$

and it follows from the implicit function theorem that there exists a function $h(x)$ with $h(0) = -1$ such that:

$$1 + xh(x) = c\sqrt{1 + 2x^2 + h(x)^2 + x^2h(x)^2}$$

Then $\phi_y(x, 0) = h(x)$ satisfies the required compatibility condition for the initial condition.

We can integrate the partial differential equation in order to obtain $\phi(x, y)$ around the origin, with the above initial condition and choice of c, if the origin is a non characteristic point; as the characteristic vector field is:

$$\dot{x} = \frac{\partial F}{\partial p}, \quad \dot{y} = \frac{\partial F}{\partial q}, \quad \dot{p} = -\frac{\partial F}{\partial x}, \quad \dot{q} = -\frac{\partial F}{\partial y}$$

where:

$$p = \phi_x, \quad q = \phi_y$$

and

$$F(x, y, p, q) = 1 - yp + xq - c\sqrt{1 + p^2 + q^2 + x^2 + y^2 + (yp - xq)^2}$$

the required transversality condition is satisfied:

$$\frac{\partial F}{\partial q}(0, 0, 0, -1) = -c\frac{-1}{\sqrt{1 + (-1)^2}} = \frac{1}{2} \neq 0$$

The Gauss map of the surface S just constructed is non degenerate around the origin, as Γ_1 is an immersion:

$$\Gamma_1 = \frac{1}{\sqrt{1 + \phi_x^2 + \phi_y^2 + x^2 + y^2 + (y\phi_x - x\phi_y)^2}}(1 + y\phi_x - x\phi_y, -\phi_x + y, -\phi_y - x)$$

and at the origin $\Gamma_1 = (\sqrt{2}/2, 0, -\sqrt{2}/2)$ and:

$$\frac{\partial}{\partial x}\Gamma_1 = (\sqrt{2}, \sqrt{2}/2, -\sqrt{2}), \quad \frac{\partial}{\partial y}\Gamma_1 = (0, -\sqrt{2}/2, 0)$$

taking in account that:

$$\phi_{xx}(0,0) = -1, \quad \phi_{xy}(0,0) = h'(0) = 2, \quad \phi_{yy}(0,0) = 0$$

The value of $\phi_{yy}(0,0)$ can be computed from the fact that the characteristic vector field at $(0,0,0,-1)$ is:

$$(F_p, F_q, -F_x, -F_y)|_{(0,0,0,-1)} = \left(0, \frac{1}{2}, 1, 0\right)$$

The surface S has Gauss curvature and normal curvature that coincide, as $0 = \Gamma_2^* \sigma = (K - \kappa)\, \omega_1 \wedge \omega_2$, and therefore is an isoclinic surface, but it is not congruent to a Lagrangean surface as neither the image of Γ_1 nor the image of Γ_2 is contained in a great circle; this seems to contradict Theorem 1 in [2].

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